Lower deviation probabilities for branching random walks

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Joint work with Xinxin Chen (陈昕昕，Lyon 1)

Content

- Branching Brownian motion
- Maximum
- Branching random walk
- Main results

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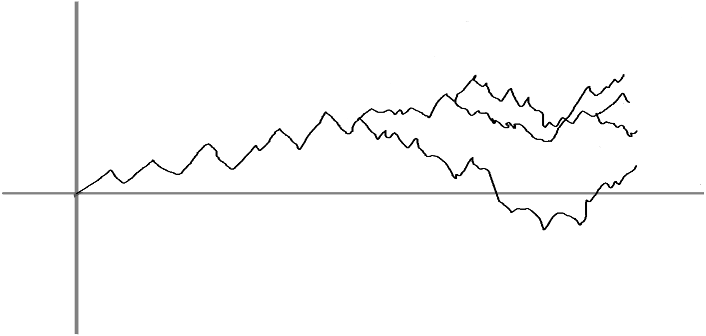

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- Bramson (1983): $M_t - (\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t)$ converge in law.

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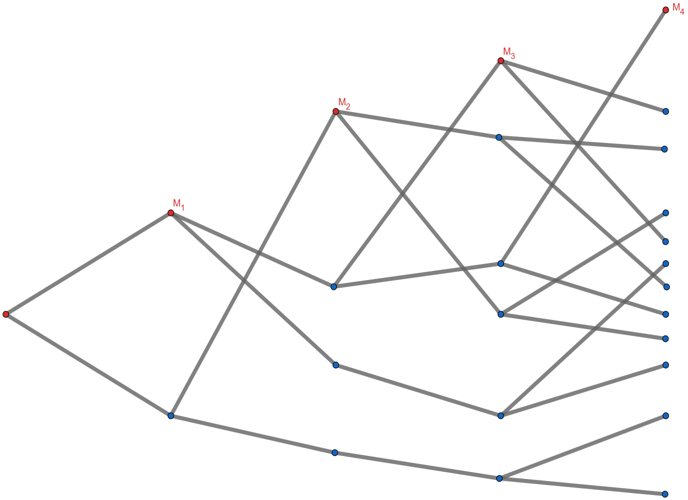

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- Chen and He (2019): Lower deviation
  $\mathbb{P}(M_n \leq x^*n - y^* \log n - \ell_n), \ell_n = O(n)$.

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- Hu (2016) studied the case of $\ell_n = o(\log n)$.

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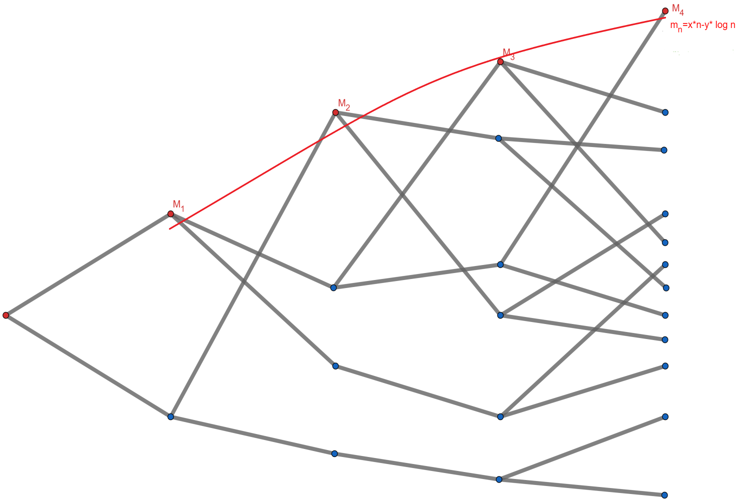
Aïdékon (2013, AOP): weak convergence.

$\mathbb{P}(M_n \leq x^* n - y^* \log n + x)$



Gantert and Höfelsauer (2018, ECP): large deviation

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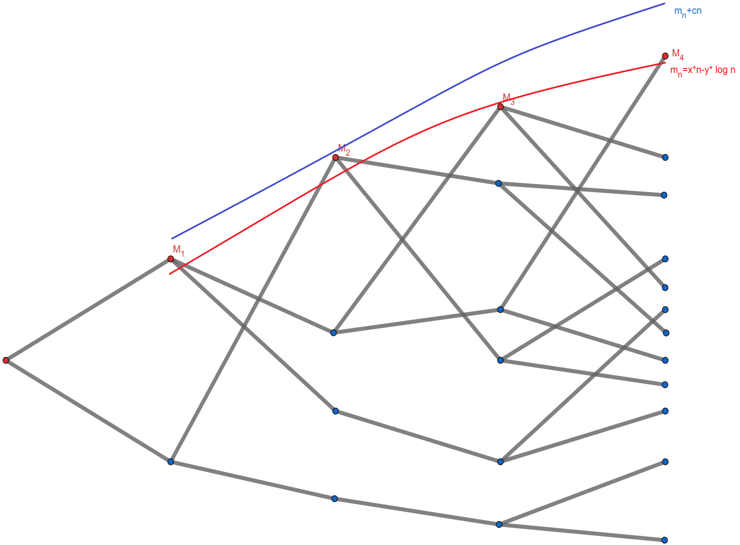

Chen and He (2019+, AIHP): Lower deviation

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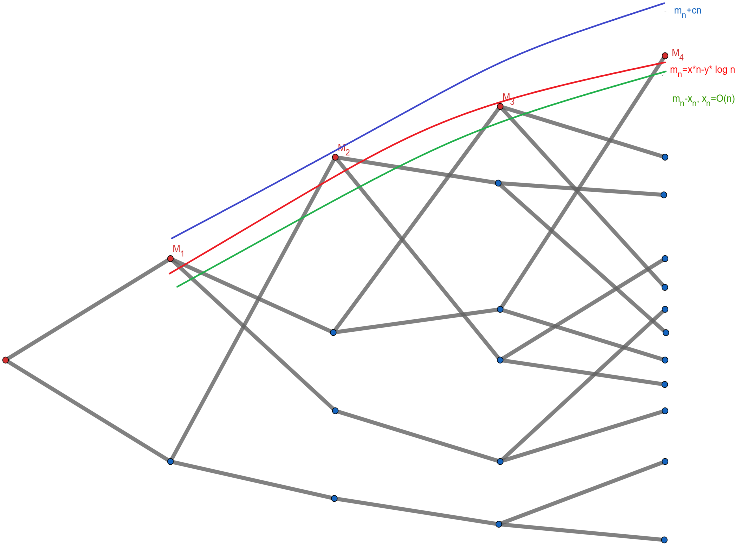

Thanks!